same relation as Eq. (24) also holds for the displacement and momentum thicknesses. Since the wall friction and heat transfer are proportional to $f''(0)$, it can be shown that the values for the conical N-S equations are larger than for the complete N-S equations by the factor of $2/\sqrt{3} \cong 1.154$, i.e., by about 15%.

Concluding Remarks

The axisymmetric viscous flows for the conical Navier-Stokes (N-S) equations and the complete N-S equations look very much the same; the boundary layers both being described essentially by the Blasius equation. The similarity scales, however, are different; hence, the boundary layers for the conical N-S equations are thinner. Because the displacement thickness is smaller for the conical N-S equations, the shock location will be displaced outward by a lesser amount than for the complete N-S equations when viscous interaction effects are taken into account, i.e., when ϵ becomes larger (or r smaller). This discrepancy in the shock location, compared with experiment, has also been noticed in Refs. 2 and 3.

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Numerical Evaluation of the Velocities Induced by Trailing Helical Vortices

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Nomenclature

I_u, I_v, I_w = integrals that determine U_i, V_i and W_i, see Eq. (1)
$\mathcal{I}_u, \mathcal{I}_v, \mathcal{I}_w$ = integrands defined in Eqs. (1-3)
I_{ur}, I_{vr}, I_{wr} = remainders used in determining I_u, I_v, and I_w
n = number of blades
p = vortex pitch
$R, R(j,n)$ = lengths defined in Eqs. (4) and (1), respectively
r = radius
U, V, W = velocity in direction of x,y,z in Fig. 1
U_i, V_i, W_i = velocities induced by the trailing helical vortices
x,y,z = co-ordinate directions defined in Fig. 1
β = defined in Eq. (8)
ϵ = absolute error in approximating I_u, I_v, and I_w using remainders
Γ = vortex strength divided by tip radius
θ = angle defined in Fig. 1
θ_m = upper limit for finite integrals

Subscripts
i = point at which induced velocities are required
v = trailing vortex

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Introduction

BOUNDARY integral or "panel" analyses of rotors, propellers, and wind turbines all require the velocities induced by the trailing helical vortices; often these are assumed to have constant pitch. A straight-forward application of the Biot-Savart law then gives U_i, V_i, and W_i in terms of infinite integrals which have no analytic solution. For example, in the co-ordinate system defined in Fig. 1, the axial velocity induced by the n vortices (all of radius r_v) trailing from n blades is given by

$$U_i = \frac{\Gamma}{4\pi} I_u(n) = \frac{\Gamma}{4\pi}\int_0^\infty \mathcal{I}_u(n)\, d\theta$$

where

$$\mathcal{I}_u(n) = \sum_{j=0}^{n-1} \frac{r_v^2 - r_i r_v \cos(\theta + \theta_v + 2\pi j/n - \theta_i)}{R^3(j,n)} \tag{1}$$

and

$$R^2(j,n) = r_i^2 + r_v^2 - 2r_i r_v \cos(\theta + \theta_v + 2\pi j/n - \theta_i) + (p\theta + x_v - x_i)^2$$

Only one blade and one trailing vortex are shown in Fig. 1 and all lengths are normalized by the tip radius. The V_i and W_i are obtained from Eq. (1) by replacing $\mathcal{I}_u(n)$ by $\mathcal{I}_v(n)$ and $\mathcal{I}_w(n)$, respectively, where

$$\mathcal{I}_v(n) = \sum_{j=0}^{n-1}\left[\frac{pz_i - pr_v\cos(\theta + \theta_v + 2\pi j/n)}{R^3(j,n)} - \frac{(p\theta + x_v - x_i) r_v \sin(\theta + \theta_v + 2\pi j/n)}{R^3(j,n)}\right] \tag{2}$$

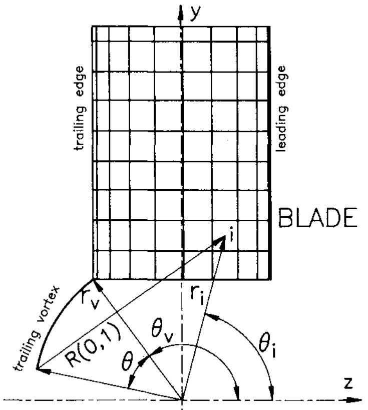

Fig. 1 Coordinate system for wind turbine. View is downstream at blade; the axial or x direction is into page. Typical distribution of panels on lower surface is shown; blade is rotating clockwise.

$$I_w(n) = \sum_{j=0}^{n-1} \frac{-py_i + pr_v \sin(\theta + \theta_v + 2\pi j/n)}{R^3(j,n)}$$

$$\frac{-(p\theta + x_v - x_i)r_v \cos(\theta + \theta_v + 2\pi j/n)}{R^3(j,n)} \tag{3}$$

In the method developed by Rand and Rosen,[1] Graber and Rosen[2] and Chiu and Peters,[3] each of $I_u(n)$, $I_v(n)$, and $I_w(n)$ is approximated as the sum of a finite integral with an upper limit θ_m and an analytic "remainder" covering the range θ_m to infinity. The finite integral is evaluated numerically, and the remainder is the integral of a mean of the so-called upper and lower "bounds" for $I_u(n)$, $I_v(n)$ and $I_w(n)$. The common alternative to the use of remainders is some form of "far-wake" approximation, which also gives analytical results. Examples are given by Miller[4] and Hess and Valerazo,[5] but these are more complicated than the remainders derived here.

There are two unsatisfactory aspects of the method of Refs. 1-3. First, there is no guarantee that the bounds are, in practice, tight upper and lower bounds on $I_u(n)$, $I_v(n)$, and $I_w(n)$. Second, the determination of the mean necessarily involves uncertainty over the relative contributions from the upper and lower bounds. While these are not necessarily major problems, they can be avoided by using alternative remainders that are integrals of the "semiasymptotic" forms of $I_u(n)$, $I_v(n)$, and $I_w(n)$. The alternative remainders are simpler than those in Refs. 1-3 but appear to be useful only when p is larger than for the rotors considered in those references. The suitable range of n has not been determined, but the method seems particularly appropriate for $n = 2$ because then the integrands in Eqs. (1-3) oscillate about the semi-asymptotic approximations.

Derivation of Alternative Remainders

The derivation of the alternative remainders, I_{ur}, I_{vr}, and I_{wr}, is straightforward. First we chose a value of θ, θ_m, for instance, large enough for $R^2(j,n)$ in Eqs. (1-3) to be replaced by R^2 for $\theta > \theta_m$, where

$$R^2 = r_i^2 + r_v^2 + (p\theta + x_v - x_i)^2 \tag{4}$$

[The asymptotic form of $R^2(j,n)$ is just $(p\theta)^2$ so that Eq. (4) is the semiasymptotic form.] This equalizes the denominators in Eqs. (1-3), and $I_u(n)$, for example, can be approximated as

$$\frac{r_v}{R^3}\left[nr_v - r_i \sum_{j=0}^{n-1} \cos(\theta + \theta_v + 2\pi j/n - \theta_i)\right] \tag{5}$$

$$= nr_v^2/R^3 \tag{6}$$

because

$$\sum_{j=0}^{n-1} \cos(\theta + \theta_v + 2\pi j/n - \theta_i) = 0 \tag{7}$$

for *any* n 1. The integration of Eq. (6) between θ_m and infinity gives

$$I_{ur}(n) = nr_v^2\beta/p \tag{8}$$

where

$$\beta = [1 - (p\theta_m + x_v - x_i)/R(\theta_m)]/[r_v^2 + r_i^2]$$

A similar use of Eq. (7) for the other components gives, upon integration,

$$I_{vr}(n) = nz_i\beta \tag{9}$$

$$I_{wr}(n) = -ny_i\beta \tag{10}$$

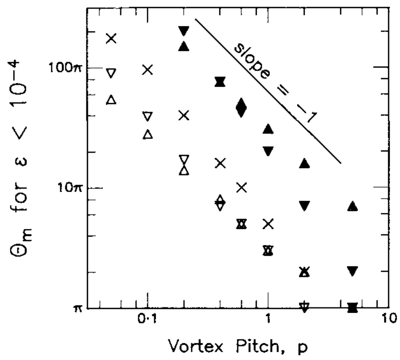

Fig. 2 Variation of the required θ_m with p; $n = 2$, $x_i = x_v = 0$, $\theta_i = \theta_v = \pi/2$, $r_i = 0.9$, and $r_v = 0.6$ $I_u(2)$, ∇; $I_w(2)$, Δ; $I_v(2)$, X; open symbols are with remainders; closed symbols without remainders.

respectively. Equations (8-10) are considerably simpler than Eqs. (11-18) of Chiu and Peters.[3] (Of course, all of these equations have the correct asymptotic form.) To return to the example, $I_u(n)$ can now be approximated as

$$I_u(n) \simeq \int_0^{\theta_m} I_u(n)\, d\theta + I_{ur}(n) \tag{11}$$

Discussion and Conclusions

The efficacy of using Eq. (11) and its equivalents for $I_v(n)$ and $I_w(n)$ can be investigated using $n = 2$, $x_i = x_v = 0$, $\theta_i = \theta_v \pi/2$, $r_i = 0.9$, and $r_v = 0.6$. This was the case considered by Rand and Rosen[1] and Chiu and Peters[3] with $p = 0.05$; the present results were obtained for $0.05 \leq p \leq 5.0$. The θ_m (and hence $\theta_m + \theta_v - \theta_i$) was set to $m\pi$ where $1 \leq m \leq 200$ and an integer, and the finite integral was evaluated using Romberg integration with an absolute error of 10^{-6}. The error, ϵ, in using the remainders was obtained from the difference between Eq. (11) (or its equivalents) for the particular θ_m and that for $\theta_m = 200\pi$. The results are shown in Fig. 2 in terms of the value of θ_m required to make ϵ less than 10^{-4}. This value of ϵ was used by Chiu and Peters[3] and was chosen here to allow comparison with their results. Further, it is sufficiently larger than the errors in determining the finite integrals not to interact with those errors. Figure 2 also shows θ_m required for the same ϵ *without* using the remainders. For $I_u(2)$ and $I_w(2)$, the remainders reduce the required θ_m by an order of magnitude. By Eq. (9), $I_{vr}(2)$ is zero whenever $z_i = 0$, as in this case.

For all components, the required θ_m depends strongly on p; when $p = 0.05$, θ_m is 90, 176, and 55π for $I_u(2)$, $I_v(2)$, and $I_w(2)$, respectively. Figures 4 and 6 in Ref. 3 show that the same accuracy was achieved at around 15π for $p = 0.05$ using their remainders. The present method, therefore, is unsuitable for small values of p. However, as p increases to values more typical of wind turbines and propellers, that is, around 0.4 and greater, the present remainders become more attractive. For $I_u(2)$ and $I_w(2)$, the required θ_m is only 3π when $p = 1.0$. As shown in Fig. 2, the required θ_m is nearly inversely proportional to p. In other words, the streamwise distance to the start of the region where the semiasymptotic approximation is valid is almost independent of p. Since this region can be associated with the farwake, the present remainders could be used as the basis of a very simple far-wake model.

The present method is least satisfactory for $I_v(2)$. However, there are two reasons why V_i need not be evaluated as accurately as U_i and W_i. First, the y axis is nearly tangential to most panels on the blades so that the y-direction velocity, of which V_i is a component, has little effect in determining the singularity distribution. Second and subsequently, the component of the tangential velocity due to the y-direction velocity is usually much smaller than the other components so that it does not contribute significantly to the blades' surface pressure.

It was stated earlier that the present method is particularly suited for $n = 2$. The reason is that $I_u(2)$ and $I_{ur}(2)$ are equal whenever $\theta_m + \theta_v - \theta_i$ is an odd-integer multiple of $\pi/2$, as are $I_v(2)$ and $I_{vr}(2)$ and $I_w(2)$ and $I_{wr}(2)$. The semiasymptotic approximations are then, in some sense, a mean of the actual oscillating integrands. They lead to remainders which have the advantage of being simpler than those in Refs. 1-3 but have the disadvantage of being useful over a smaller range of vortex pitch.

Acknowledgment

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Higher Order Sensitivity Analysis of Complex, Coupled Systems

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Introduction

IN design of engineering systems, the "what if" questions often arise such as, What will be the change of the aircraft payload if the wing aspect ratio is incremented by 10%? Answers to such questions are commonly sought by incrementing the pertinent variable and re-evaluating the major disciplinary analyses involved. These analyses are contributed by engineering disciplines that are usually coupled, as are the aerodynamics, structures, and performance in the context of the preceding question.

The "what if" questions may be answered to the linear order of approximation without using the "increment-and-reevaluate" approach and without finite differencing of the system analysis, by using a method introduced in Ref. 1 for calculating the first derivatives of behavior of coupled systems with respect to design variables. The method called the system sensitivity analysis has been demonstrated in formal optimization that used the derivatives to guide search in multidimensional design space, e.g., Refs. 2 and 3. Obviously, if the problem is strongly nonlinear, the efficiency of such search will improve if second- and, possibly, higher-order derivatives are available to the search algorithm. A need for the second-order derivatives to enhance the method of Ref. 1 applied in such problems is discussed in Ref. 4.

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This Note extends the algorithm of Ref. 1 to include the derivatives of the second and higher orders, again, without finite differencing of the system analysis. It achieves that by recursive application of the same implicit function theorem that underlies Ref. 1. As a supplement to Ref. 1, the Note is not self contained; it references equations in Ref. 1 (such references are shown as "Ref. 1/Eq. No.") and requires that reference as a prerequisite.

First-Order Sensitivity Analysis

The sensitivity problem stated in Ref. 1 calls for calculation of the derivatives of a vector Y solving the governing equations, Ref. 1/Eq. (1), with respect to a design variable X_k. The algorithm developed in Ref. 1 yields the derivative of Y as a solution vector Z of the sensitivity equations, which are linear, simultaneous, algebraic equations of the form

$$AZ = R \tag{1}$$

In Ref. 1, the terms in the preceding equation are defined by two equivalent sets of equations: either Ref. 1/Eq. (4) (based on the residuals and called the Global Sensitivity Equations 1, GSE1) or Ref. 1/Eq. (8) (based on the output/input partial sensitivity derivatives and called the Global Sensitivity Equations 2, GSE2). The contents of the matrix of coefficients A and the right-hand-side vector R are different in the preceding two alternative formulations, but the solution vector Z is the same and represents the derivative of Y with respect to the kth element of the vector of design variables X. It has the meaning of the total derivative because it reflects both the direct and indirect influences of X_k on Y.

Sensitivity Analysis of Second and Higher Orders

Generalization of the preceding first-order sensitivity analysis to higher orders is straightforward by taking advantage of the linearity of Eq. (1) herein. Although most of the linear algebra texts ignore the matter, algorithms for sensitivity analysis of the linear algebraic equations solution have been developed in structural sensitivity analysis. They stem from the same implicit function theorem that was the basis for Ref. 1, e.g., Refs. 5 and 6, and they extend to the second-order derivatives, e.g., Refs. 7 and 8. The pattern established in these algorithms will be adapted to solve the problem at hand.

A compact notation for the derivatives will be used in the remainder of the Note:

$$(\;)^q_{klm\cdots} = \partial^q(\;)\,\partial X_k\,\partial X_l\,\partial X_m \cdots \tag{2}$$

where any subscript may by repeated as required to form a high-order, mixed derivative with respect to any combination of variables X.

In the preceding notation, the correspondence of Eq. (1) above to Ref. 1/Eq. (4) and Ref. 1/Eq. (8) makes the derivatives of Z with respect to X equivalent to the derivatives of Y with respect to X, as follows (Z is already the first derivative of Y):

$$\begin{aligned} Z^0 &= Y^1{}_k \\ Z^1{}_l &= Y^2{}_{kl} \\ Z^2{}_{lm} &= Y^3{}_{klm} \\ &\;\;\vdots \\ Z^N_{lm\cdots} &= Y^{N+1}{}_{klm\cdots} \end{aligned} \tag{3}$$

Repeated differentiation of Eq. (1) yields the derivatives of Z according to a regular pattern, shown up to the fourth derivative as

$$AZ^1_l = R^1_l - A^1_l Z^0 \tag{4}$$

$$AZ^2_{lm} = R^2_{lm} - A^1_m Z^1_l - D^1_m (A^1_l Z^0) \tag{5}$$

$$AZ^3_{lmn} = R^3_{lmn} - A^1_n Z^2_{lm} - D^1_n (A^1_m Z^1_l) - D^2_{mn} (A^1_l Z^0) \tag{6}$$